Int J Theor Phys (2008) 47: 1809–1816
DOI 10.1007/s10773-007-9623-8

A Closed-System Approach to Quantum Retrodiction in Open Systems

Andrew J. Scroggie · John Jeffers

Received: 1 October 2007 / Accepted: 27 November 2007 / Published online: 4 December 2007

Abstract Ban (Int. J. Theor. Phys. 46:184, 2007) has shown how retrodictive open systems evolution may be treated as unitary using non-equilibrium thermo field dynamics. Here we describe the application of another technique with the same purpose, Fano diagonalisation.

Keywords Retrodiction · Open systems · Fano diagonalisation

1 Introduction

Retrodictive quantum theory, in which the state of the system is assigned on the basis of the measured state, has a long history [1–4, 26, 29]. The recent upsurge in interest in the topic resulted from the relation of the predictive and retrodictive states by Bayes' theorem [12, 24, 25]. The main use of retrodictive quantum theory is to simply retrodict the prepared state given the result of a measurement, and thus it is especially useful in quantum communication schemes [15], and also in the calculation of measurement operators in coupled systems [22, 28].

In closed systems the time evolution of the system between preparation and measurement is easy to account for both forwards and backwards in time via the unitary time translation operator. Either the prepared state $\hat{\rho}^{\mathrm{p}}$ evolves forward in time according to

$$\hat{\rho}^{\mathrm{p}}(t_{\mathrm{m}}) = \hat{U}(t_{\mathrm{m}} - t_{\mathrm{p}})\hat{\rho}^{\mathrm{p}}(t_{\mathrm{p}})\hat{U}^{\dagger}(t_{\mathrm{m}} - t_{\mathrm{p}}), \tag{1}$$

or the retrodictive or measured state $\hat{\rho}^{\mathrm{r}}$ evolves backwards in time via

$$\hat{\rho}^{\mathrm{r}}(t_{\mathrm{p}}) = \hat{U}^{\dagger}(t_{\mathrm{m}} - t_{\mathrm{p}})\hat{\rho}^{\mathrm{r}}(t_{\mathrm{m}})\hat{U}(t_{\mathrm{m}} - t_{\mathrm{p}}), \tag{2}$$

A.J. Scroggie · J. Jeffers (✉)
Department of Physics, SUPA, University of Strathclyde, John Anderson Building, 107 Rottenrow, Glasgow G4 0NG, UK
e-mail: john@phys.strath.ac.uk

where t_p, t_m are the preparation and measurement times. $\hat{\rho}^\mathrm{r}$ is related to the probability operator $\hat{\pi}_l$ associated with the measurement result l via [17]

$$\hat{\rho}^\mathrm{r}(t_\mathrm{m}) = \hat{\pi}_l / \operatorname{Tr} \hat{\pi}_l. \tag{3}$$

Calculated probabilities and expectation values are unaffected by the choice of predictive or retrodictive description of the system.

In open systems, however, the evolution forwards in time is normally described by a master equation which must be of Lindblad form

$$\dot{\hat{\rho}} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}] + \sum_i (2\hat{A}_i \hat{\rho} \hat{A}_i^\dagger - \hat{A}_i^\dagger \hat{A}_i \hat{\rho} - \hat{\rho} \hat{A}_i^\dagger \hat{A}_i), \tag{4}$$

where $\hat{A}_i$ is a system operator. Retrodictive open systems evolution is a little more difficult to describe. It is possible to write down a retrodictive master equation for the system [15],

$$\dot{\hat{\rho}}^\mathrm{r} = \frac{i}{\hbar}[\hat{\rho}^\mathrm{r}, \hat{H}] - \sum_i (2\hat{A}_i^\dagger \hat{\rho}^\mathrm{r} \hat{A}_i - \hat{A}_i \hat{A}_i^\dagger \hat{\rho}^\mathrm{r} - \hat{\rho}^\mathrm{r} \hat{A}_i \hat{A}_i^\dagger) - 2\hat{\rho}^\mathrm{r} \operatorname{Tr}\left(\hat{\rho}^\mathrm{r} \sum_i \hat{A}_i^\dagger \hat{A}_i \right), \tag{5}$$

which provides the evolution of the measured state backwards in time (dot denotes forward time derivative), but as can be seen this equation is in general nonlinear in the retrodictive density operator. A slightly easier technique is to follow the evolution of the probability operator associated with the measurement, which is given by the solution to a linear master equation [15]

$$\dot{\hat{\pi}} = \frac{i}{\hbar}[\hat{\pi}, \hat{H}] - \sum_i (2\hat{A}_i^\dagger \hat{\pi} \hat{A}_i - \hat{A}_i \hat{A}_i^\dagger \hat{\pi} - \hat{\pi} \hat{A}_i \hat{A}_i^\dagger). \tag{6}$$

It is not always possible to solve master equations, either predictive or retrodictive. Even when solutions of the predictive master equation are known it is unlikely that the corresponding retrodictive equation will have a simple solution. It is sometimes possible, however, to relate the solution of a retrodictive master equation to that of a predictive one—a technique first employed for atoms [13], and later for optical amplifiers, attenuators and detectors [19]. Furthermore it has been shown that optical amplifiers and attenuators are predictive/retrodictive inverses: the transformation that an attenuator makes on a measured state backwards in time is the same as that made by an amplifier on a prepared state forwards in time, provided that the amplifier gain is the reciprocal of the attenuator loss [14, 19]. For such a system one never needs to solve a retrodictive master equation.

It would clearly be easier to treat open systems in a similar manner to closed systems, with a simple Hamiltonian-type evolution. This was the topic of a recent paper by Ban [9], which showed that it was possible to formulate retrodictive open systems evolution in the same way as for a closed system using a technique known as non-equilibrium thermo field dynamics [5–8]. In this method a state in the system Hilbert space is mapped onto one in a tensor product of the system space and an auxiliary Hilbert space of the same dimension. Expectation values, and both forward and reverse-time evolution generators can be found, and it is this which allows the simple closed system-type treatment.

In this paper we have the same aim as Ban—to treat retrodictive open systems as if they were closed, thus avoiding the more difficult semigroup evolution. In order to do this, however, we employ a different technique, Fano diagonalisation [11, 16], described in the next section. In Sect. 3 we apply Fano theory to retrodictive evolution by means of an example,

that of a leaky optical cavity, which provides a good model for a simple lossy system. The extension to other systems which can be diagonalised by the Fano technique is straightforward. We then conclude and summarise our results.

2 Fano Diagonalisation

Fano diagonalisation [16] is a technique used to obtain the dressed states of a coupled system [10, 11]. The technique is widely applicable and can also be employed to calculate dressed operators [11] in lossy linear dielectrics [18], nonlinear dielectrics [27], and for atom lasers [21]. The usefulness of the technique in prediction and, as we shall see, in retrodiction, rests on simplicity of the evolution of the dressed operators.

We proceed by example and apply the theory to the simple system of a single mode of the electromagnetic field coupled to a continuum of external modes. The diagonalisation is performed elsewhere [11], but we repeat the main results here for clarity. The Hamiltonian of the system is

$$\hat{H} = \hbar\omega_0 \sum_{n=0}^{\infty} n|n\rangle\langle n| + \hbar \int d\Delta \Delta |\Delta\rangle\langle\Delta| + \hbar \sum_{n=0}^{\infty} \int d\Delta (W_n |n\rangle\langle\Delta| + W_n^* |\Delta\rangle\langle n|), \quad (7)$$

where ω_0 is the frequency of the field, $|n\rangle$ is the n-photon number state, $|\Delta\rangle$ is a state of energy $\hbar\Delta$, and W_n is the interaction strength. We will make the simplifying assumption that this is independent of frequency. The generalisation to a frequency-dependent coupling is easily accommodated [11]. The goal is to find eigenstates $|\omega\rangle$ of the coupled system which satisfy the Schrödinger equation,

$$\hat{H}|\omega\rangle = \hbar\omega|\omega\rangle. \quad (8)$$

The eigenstates are written as a superposition of the eigenstates of the two systems

$$|\omega\rangle = \sum_n \alpha_n(\omega)|n\rangle + \int d\Delta \beta(\omega, \Delta)|\Delta\rangle. \quad (9)$$

Substitution of this into the Schrödinger equation gives the following pair of equations

$$n\omega_0\alpha_n(\omega) + \int d\Delta W_n \beta(\omega, \Delta) = \omega\alpha_n(\omega), \quad (10)$$

$$\sum_n W_n^* \alpha_n(\omega) + \Delta\beta(\omega, \Delta) = \omega\beta(\omega, \Delta). \quad (11)$$

We can formally solve (11) by writing

$$\beta(\omega, \Delta) = \left(\frac{\mathcal{P}}{\omega - \Delta} + z(\omega)\delta(\omega - \Delta) \right) \sum_n W_n^* \alpha_n(\omega), \quad (12)$$

where $\mathcal{P}$ stands for principal part and z is a function to be determined. This is substituted into (10) to give

$$(\omega - n\omega_0)\alpha_n(\omega) = z(\omega) W_n \sum_m W_m^* \alpha_m(\omega), \quad (13)$$

where the principle part integral vanishes. We can multiply this equation by $(\omega - n\omega_0)$ and sum over n, which results in a cancellation of summations leaving

$$z(\omega)\sum_n \frac{|W_n|^2}{\omega - n\omega_0} = 1. \tag{14}$$

At this point we have an expression for β in terms of the α_n and z, and an expression for z in terms of the coupling strength. We now impose orthogonality on the diagonalised system in order to fix the α_n. From (9),

$$\langle\omega'|\omega\rangle = \delta(\omega - \omega') = \sum_n \alpha_n^*(\omega')\alpha_n(\omega) + \int d\Delta\beta^*(\omega', \Delta)\beta(\omega, \Delta). \tag{15}$$

We substitute the expression for α_n in terms of z, (13), and (11), and resolve the subsequent product of principal parts as

$$\frac{\mathcal{P}}{\omega - \Delta}\frac{\mathcal{P}}{\omega' - \Delta} = \frac{\mathcal{P}}{\omega' - \omega}\left(\frac{\mathcal{P}}{\omega - \Delta} - \frac{\mathcal{P}}{\omega' - \Delta}\right) + \pi^2\delta(\omega - \Delta)\delta(\omega' - \Delta). \tag{16}$$

Some simple algebra then gives

$$\sum_n \alpha_n(\omega)W_n^* = \frac{1}{\pi + iz(\omega)}, \tag{17}$$

where we have chosen an arbitrary phase. This expression can be inserted into (13) to give

$$\alpha_n(\omega) = \frac{W_n z(\omega)}{(\omega - n\omega_0)[\pi + iz(\omega)]}. \tag{18}$$

The eigenstates of the system $|\omega\rangle$ have the simple $\exp(-i\omega t)$ time dependence. However, we seek the evolution of the bare states $|n\rangle$. These can be written as an integral over the set of dressed states

$$|n\rangle = \int d\omega f(\omega)|\omega\rangle, \tag{19}$$

so that $\langle\omega|n\rangle = f(\omega) = \alpha_n^*(\omega)$ using (9). The number states then have the simple time evolution

$$|n\rangle(t) = \int d\omega\alpha_n^*(\omega)\exp(-i\omega t)|\omega\rangle. \tag{20}$$

3 Retrodiction

Normally we think of a system being prepared in a particular state and evolving forwards in time, and so the time is positive in (20). The equation can be used to derive the probability that the system would be found in a particular state at some time after the preparation. For example, if the system is prepared in the state $|n\rangle$ at the preparation time t_p, the predictive conditional probability $P^\mathrm{p}(l|n)$ that it would be measured in the state $|l\rangle$ at the measurement time t_m is given by

$$P^\mathrm{p}(l|n) = \left|\int d\omega\alpha_l(\omega)\alpha_n^*(\omega)\exp[-i\omega(t_\mathrm{m} - t_\mathrm{p})]\right|^2. \tag{21}$$

This set of probabilities can be straightforwardly used to construct the density operator of the system at all times after the preparation. If the system is initially prepared in a superposition of the number states, then we simply use a superposition of two different evolved number states. Although in this example the measurement and preparation correspond to pure states, for the sake of generality and to facilitate later use it is useful to proceed a little more formally, and so we write the prepared state as a density operator $\hat{\rho}_n = |n\rangle\langle n|$, the evolution of which is found from (20). The measurement is described by a probability operator measure [17], $\hat{\pi} = \sum_l \hat{\pi}_l$, each of whose elements (probability operators) is in this case an orthogonal number state projector $\hat{\pi}_l = |l\rangle\langle l|$. Then we can write (21) in terms of the evolved density operator and the probability operator as $P^{\mathrm{p}}(l|n) = \mathrm{Tr}(\hat{\rho}_n(t_{\mathrm{m}})\hat{\pi}_l)$.

Equation (20) is, however, equally valid for negative times. If the system is measured in state $|l\rangle$ we can use the equation to evolve the probability operator back in time to the preparation

$$\hat{\pi}_l(t_{\mathrm{p}}) = \int d\omega \int d\omega' \alpha_l^*(\omega) \exp[-i\omega(t_{\mathrm{p}} - t_{\mathrm{m}})]|\omega\rangle\langle\omega'| \exp[i\omega'(t_{\mathrm{p}} - t_{\mathrm{m}})]\alpha_l(\omega'). \quad (22)$$

We can write the predictive conditional probability that measurement result l is found as

$$P^{\mathrm{p}}(l|n) = \mathrm{Tr}(\hat{\rho}_n\hat{\pi}_l(t_{\mathrm{p}})) = \left| \int d\omega \alpha_n(\omega)\alpha_l^*(\omega) \exp[-i\omega(t_{\mathrm{p}} - t_{\mathrm{m}})] \right|^2, \quad (23)$$

which is identical in form to (21), a consequence of the fact that the probabilities which make up the diagonal elements of the forward-evolved predictive density operator must be identical to the equivalent weights in the backward-evolved probability operator (see Appendix in [20]).

The main advantage in using retrodictive quantum theory lies in the calculation of retrodictive conditional probabilities that particular states were prepared given the results of measurements. The theory is particularly useful in situations where the state preparation device is unbiased [12], which means that each state from an orthogonal set spanning the system space is equally likely to be "prepared".[1] The retrodictive density operator then provides most information about the system. For example, if the measurement results correspond to number states the retrodictive state is the appropriately normalised version of (22). If the preparation device is unbiased and prepares number states, we can find the retrodictive conditional probability that the system was prepared in the number state $|n\rangle$ by evolving the normalised version of (22) for the appropriate length of time

$$P^{\mathrm{r}}(n|l) = \langle n|\hat{\rho}_l^{\mathrm{r}}(t_{\mathrm{p}})|n\rangle. \quad (24)$$

This has a considerable advantage over using Bayes' theorem, as we do not have to calculate all of the predictive conditional probabilities, only the single retrodictive one that we are interested in. As a simple example we show in Fig. 1 a plot of the retrodictive conditional probability that the n-photon number state was prepared prior to one photon being recorded in the cavity, as a function of evolution time, for equal prior probabilities of all photon number states between 1 and 5. The plot shows the physically-reasonable result that if the state was prepared a long time before measurement, and one photon was measured, that state $|n\rangle$ is n times as likely to have been prepared as state $|1\rangle$. If one photon is recorded a

[1] Strictly speaking we mean by this that each state from an orthogonal set spanning the space of the system is equally likely to be part of the prepared state, prior to one of the actual prepared states being selected [23].

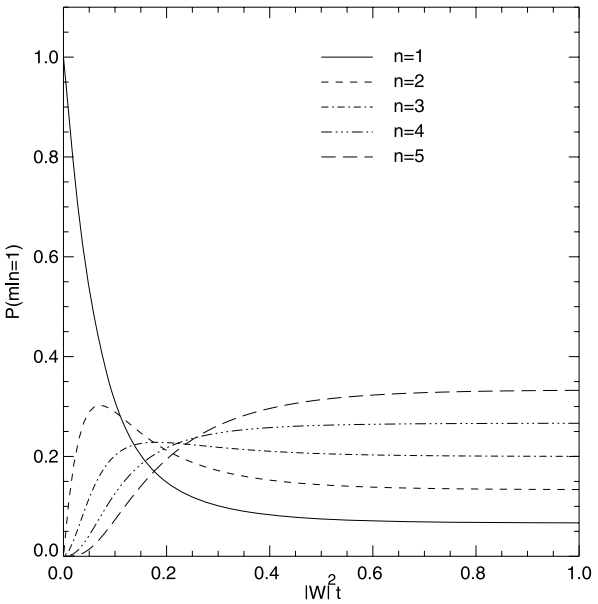

Fig. 1 Retrodictive conditional probability that the n-photon number state was prepared given one photon being recorded later, against evolution time. Here the coupling strength W is independent of n

very short time after preparation it is overwhelmingly-likely that the one-photon state was prepared.

The same mathematics applies to superpositions of number states. A simple example is the coherent state. The evolution of any superposition state $|\psi\rangle = \sum_n a_n|n\rangle$ is simply given by a superposition of the evolutions of the number states

$$|\psi\rangle(t) = \sum_n \int d\omega \alpha_n^*(\omega) \exp(-i\omega t)|\omega\rangle. \tag{25}$$

Suppose that the cavity is prepared with equal prior probability in coherent states of mean photon numbers $|\delta|^2 = 1..4$.[2] The state of the cavity is later measured in the photon number basis. Figure 2 shows the retrodictive conditional probability that each of the coherent states was prepared given that the two-photon state was later measured. If the state is measured immediately after preparation the probabilities reflect the 2-photon number state amplitudes in the particular coherent states. If there is a long time between preparation and measurement there is a large disparity between the probabilities that different coherent states were prepared. This reflects the fact that coherent states of a higher amplitude are superpositions whose high photon number state amplitudes are proportionately much larger. Detecting two photons after a long time means that it is very likely that the cavity contained a high number of photons at preparation.

The generalisation of the above calculations to mixed state preparation or measurement is also easily accomplished, all of which underlines the simplicity of the method, and its advantages over solving master equations.

[2]Whether or not this is possible is controversial. See [23] and references therein.

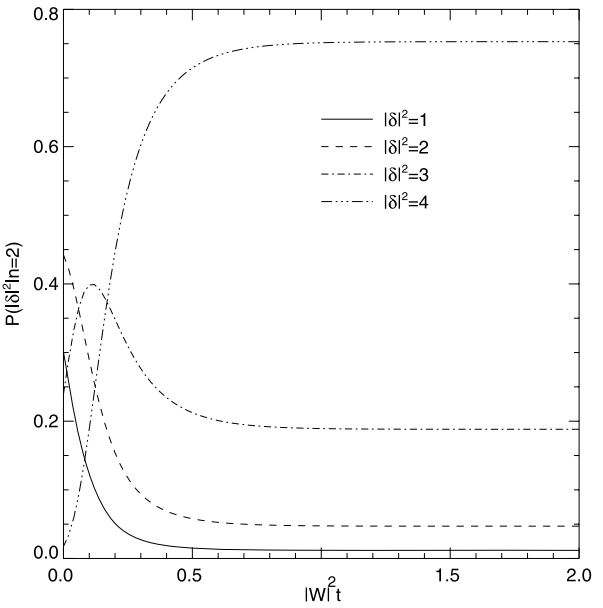

Fig. 2 Retrodictive conditional probability that the coherent state with mean photon numbers 1..4 was prepared given two photons being recorded later, against evolution time. Here the coupling strength W is independent of n

4 Conclusions

Whilst the evolution of open quantum systems is described by well-known master equations the solutions of the equations are not normally a trivial matter, except in the simplest situations. The retrodictive master equation also suffers from this problem, which is all the more serious, as open systems evolution is not time-symmetric. This difficulty was recognised and circumvented using non-equilibrium thermo field dynamics. In this way the open system can be treated as closed, and the simple Hamiltonian evolution can be used to evolve the system both predictively and retrodictively.

Here we have described a different route to simplifying retrodictive evolution by writing the open system in terms of eigenstates. Fano diagonalisation is a well-tested method for finding the eigenstates of an open system, and so, as with the earlier method, it enables the evolution of the system to be followed both forwards and backwards in time with equal ease. One caveat is that it is not always possible to find eigenstates using Fano's method, but when it is possible, as it is for many systems studied, for example, in quantum optics, the method will render the retrodictive evolution of any measured state trivial.

References

1. Aharonov, Y., Albert, D.Z.: Is the usual notion of time evolution adequate for quantum-mechanical systems? I. Phys. Rev. D **29**, 223 (1984)
2. Aharonov, Y., Albert, D.Z.: Is the usual notion of time evolution adequate for quantum-mechanical systems? II. Relativistic considerations. Phys. Rev. D **29**, 228 (1984)
3. Aharonov, Y., Vaidman, L.: Complete description of a quantum system at a given time. J. Phys. A **24**, 2315 (1991)
4. Aharonov, Y., Bergmann, P.G., Lebowitz, J.L.: Time symmetry in the quantum process of measurement. Phys. Rev. **134**, B1410 (1964)
5. Arimitsu, T.: General formulation for open systems mirror operation. J. Phys. Soc. Jpn. **51**, 1720 (1982)

6. Arimitsu, T., Umezawa, H.: A general formulation of non-equilibrium thermo field dynamics. Prog. Theor. Phys. **74**, 429 (1985)
7. Arimitsu, T., Umezawa, H.: Non-equilibrium thermo field dynamics. Prog. Theor. Phys. **77**, 32 (1987)
8. Arimitsu, T., Umezawa, H.: General structure of non-equilibrium thermo field dynamics. Prog. Theor. Phys. **77**, 53 (1987)
9. Ban, M.: Quantum retrodiction in non-equilibrium thermo field dynamics. Int. J. Theor. Phys. **46**, 184 (2007)
10. Barnett, S.M., Radmore, P.M.: Quantum theory of cavity quasimodes. Opt. Commun. **68**, 364 (1988)
11. Barnett, S.M., Radmore, P.M.: Methods in Theoretical Quantum Optics. Oxford University Press, Oxford (1997)
12. Barnett, S.M., Pegg, D.T., Jeffers, J.: Bayes' theorem and quantum retrodiction. J. Mod. Opt. **47**, 1779 (2000)
13. Barnett, S.M., Pegg, D.T., Jeffers, J., Jedrkiewicz, O.: Atomic retrodiction. J. Phys. B: At. Mol. Opt. Phys. **33**, 3047 (2000)
14. Barnett, S.M., Pegg, D.T., Jeffers, J., Jedrkiewicz, O., Loudon, R.: Retrodiction for quantum optical communications. Phys. Rev. A **62**, 022313 (2000)
15. Barnett, S.M., Pegg, D.T., Jeffers, J., Jedrkiewicz, O.: Master equation for retrodiction of quantum communication signals. Phys. Rev. Lett. **86**, 2455 (2001)
16. Fano, U.: Effects of configuration interaction on intensities and phase shifts. Phys. Rev. **124**, 1866 (1961)
17. Helstrom, C.W.: Quantum Detection and Estimation Theory. Academic Press, New York (1976)
18. Huttner, B., Barnett, S.M.: Quantization of the electromagnetic field in dielectrics. Phys. Rev. A **46**, 4306 (1992)
19. Jedrkiewicz, O., Loudon, R., Jeffers, J.: Retrodiction for optical attenuators, amplifiers and detectors. Phys. Rev. A **70**, 033805 (2004)
20. Jeffers, J.: Retrodictive fidelities for pure state postselectors. New J. Phys. **8**, 268 (2006)
21. Jeffers, J., Horak, P., Barnett, S.M., Radmore, P.M.: Bound mode of an atom laser. Phys. Rev. A **62**, 043602 (2000)
22. Jeffers, J., Barnett, S.M., Pegg, D.T.: Retrodiction as a tool for micromaser field measurements. J. Mod. Opt. **49**, 925 (2002)
23. Pegg, D.T., Jeffers, J.: Quantum state of a laser. J. Mod. Opt. **52**, 1835 (2005)
24. Pegg, D.T., Barnett, S.M., Jeffers, J.: Quantum theory of preparation and measurement. J. Mod. Opt. **49**, 913 (2002)
25. Pegg, D.T., Barnett, S.M., Jeffers, J.: Quantum retrodiction in open systems. Phys. Rev. A **66**, 022106 (2002)
26. Penfield, R.H.: More on the arrow of time. Am. J. Phys. **34**, 422 (1966)
27. Schmidt, E., Jeffers, J., Barnett, S.M., Knöll, L., Welsch, D.-G.: Quantum theory of light in nonlinear media with dispersion and absorption. J. Mod. Opt. **45**, 377 (1998)
28. Tan, E.-K., Jeffers, J., Barnett, S.M.: Field-state measurement in a micromaser using retrodictive quantum theory. Phys. Rev. A **69**, 043806 (2004)
29. Watanabe, S.: Symmetry of physical laws. Part III. Prediction and retrodiction. Rev. Mod. Phys. **27**, 179 (1955)